4601. We simplify as follows:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}\\ &= \lim_{h\to 0}\frac{\cos(x)-\cos(x+h)}{h\cos x\cos(x+h)}.\end{aligned}$$

Using a compound-angle formula, this is

$$\begin{aligned}&\lim_{h\to 0}\frac{\cos x-\cos x\cos h+\sin x\sin h}{h\cos x\cos(x+h)}\\ =&\lim_{h\to 0}\frac{1-\cos h}{h\cos(x+h)}+\frac{\sin x\sin h}{h\cos x\cos(x+h)}.\end{aligned}$$

In the first term,

$$\frac{1-\cos h}{h}\approx\frac{1-\left(1-\frac{1}{2}h^2\right)}{h}=\tfrac{1}{2}h.$$

In the second term,

$$\frac{\sin h}{h}\approx 1.$$

These small-angle approximations give

$$\frac{dy}{dx}=\lim_{h\to 0}\tfrac{1}{2}h\sec(x+h)+\tan x\sec(x+h).$$

At this point, we can take the limit. The first term tends to zero, leaving

$$\frac{dy}{dx}=\tan x\sec x,\text{ as required.}$$

4602. Dividing the total of π radians into five parts, the smaller angle is $\pi/5$ and the larger $2\pi/5$. Using the cosine rule,

$$\begin{aligned}1 &= \phi^2+\phi^2-2\phi^2\cos\tfrac{\pi}{5}\\ \Longrightarrow 1 &= \phi^2\left(2-\frac{\sqrt{5}+1}{2}\right)\\ \Longrightarrow \phi^2 &= \frac{2}{3-\sqrt{5}}\\ &= \frac{3+\sqrt{5}}{2}.\end{aligned}$$

To find the positive square root, we set up

$$\begin{aligned}\frac{3+\sqrt{5}}{2} &= (a\sqrt{5}+b)^2\\ &\equiv 5a^2+b^2+2ab\sqrt{5}.\end{aligned}$$

Equating coefficients,

$$\begin{aligned}5a^2+b^2 &= \tfrac{3}{2},\\ 2ab &= \tfrac{1}{2}.\end{aligned}$$

Solving these simultaneously, the positive rational roots are $a=b=1/2$. So, the golden triangle has sides in the ratio $1:\phi:\phi$, where

$$\phi=\frac{\sqrt{5}+1}{2},\text{ as required.}$$

4603. Separating the variables,

$$\int\frac{1}{1-y^2}\,dx=\int 1\,dx.$$

In partial fractions,

$$\begin{aligned}&\frac{1}{1-y^2}\equiv\frac{A}{1+y}+\frac{B}{1-y}\\ \Longrightarrow\ &1\equiv A(1-y)+B(1+y).\end{aligned}$$

Substituting $y=\pm 1$ gives $A,B=\frac{1}{2}$:

$$\begin{aligned}&\tfrac{1}{2}\int\frac{1}{1+y}+\frac{1}{1-y}\,dy=\int 1\,dx\\ \Longrightarrow\ &\tfrac{1}{2}\ln\left|\frac{1+y}{1-y}\right|=x+c\\ \Longrightarrow\ &\ln\left|\frac{1+y}{1-y}\right|=2x+d\\ \Longrightarrow\ &\frac{1+y}{1-y}=e^{2x+d}.\end{aligned}$$

Substituting $x=1, y=0$, we get $d=-2$. So,

$$\begin{aligned}&\frac{1+y}{1-y}=e^{2x-2}\\ \Longrightarrow\ &1+y=e^{2x-2}-ye^{2x-2}\\ \Longrightarrow\ &y(e^{2x-2}+1)=e^{2x-2}-1\\ \Longrightarrow\ &y=\frac{e^{2x-2}-1}{e^{2x-2}+1}.\end{aligned}$$

Multiplying top and bottom by e^2,

$$y=\frac{e^{2x}-e^2}{e^{2x}+e^2},\text{ as required.}$$

4604. (a) The circle must obey the symmetry of the curves, which are reflections of each other in $x=\frac{3}{4}\pi$.

(b) The derivative is $\cos x$, so the normal gradient is $-\sec x$. At point $(a,\sin a)$, the equation of the normal is

$$\begin{aligned}&y-\sin a=-\sec a(x-a)\\ \Longrightarrow\ &y=(a-x)\sec a+\sin a.\end{aligned}$$

(c) We need this normal to pass through $\left(\frac{3}{4}\pi,0\right)$. Substituting this in,

$$\begin{aligned}&0=\left(a-\tfrac{3}{4}\pi\right)\sec a+\sin a\\ \Longrightarrow\ &0=a-\tfrac{3}{4}\pi+\sin a\cos a\\ \Longrightarrow\ &\sin 2a+2a-\tfrac{3}{2}\pi=0.\end{aligned}$$

(d) The Newton-Raphson iteration is

$$a_{n+1}=a_n-\frac{\sin 2a_n+2a_n-\frac{3}{2}\pi}{2\cos 2a_n+2}.$$

Running this with $a_0=3$, we get $a_1=2.7428$ and then $a_n\to 2.7257$ (5sf).

(e) The diameter of the circle is twice the distance from $(a, \sin a)$ to $\left(\frac{3}{4}\pi, 0\right)$. This is

$$\begin{aligned} d &= 2\sqrt{\left(2.7257 - \tfrac{3}{4}\pi\right)^2 + \sin^2 2.7257} \\ &= 1.0949... \\ &\approx 1.095, \text{ as required.} \end{aligned}$$

4605. To find the probability of the three-way union, we first add the individual probabilities of the events (inclusion):

$$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

We have overcounted the two-way intersections. So, we subtract a copy of each (exclusion),

$$-\mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A).$$

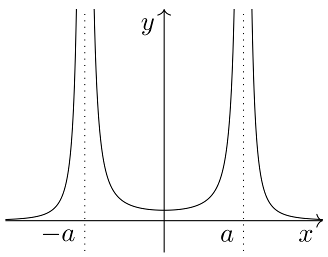
In doing so, we have oversubtracted the three-way intersection: we added it three times to begin with, then subtracted it three times, so we need to add it once more (inclusion). This gives the three-way inclusion-exclusion principle

$$\begin{aligned} &\mathbb{P}(A \cup B \cup C) \\ \equiv\ &\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ &- \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) \\ &+ \mathbb{P}(A \cap B \cap C). \end{aligned}$$

4606. Factorising the denominator,

$$\begin{aligned} y &= \frac{1}{(x^2 - a^2)^2} \\ &\equiv \frac{1}{(x-a)^2(x+a)^2}. \end{aligned}$$

This is a reciprocal quartic with a pair of double asymptotes at $x = \pm a$. It is positive for all $x \in \mathbb{R}$, and tends to 0 as $x \to \pm\infty$.



4607. The derivative is

$$\frac{dy}{dx} = kx^{k-1} - k(k+1)x^k.$$

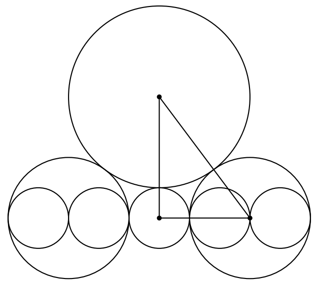
So, at $x = 1$, the gradient is $k - k(k+1) \equiv -k^2$. The y value is $1 - k$. The equation of the tangent, therefore, is

$$y - (1-k) = -k^2(x-1).$$

This crosses the x axis at $x = 3$. Subbing $(3, 0)$,

$$\begin{aligned} &-(1-k) = -k^2(3-1) \\ \Longrightarrow\ &k = -1, \tfrac{1}{2}. \end{aligned}$$

4608. By symmetry, the triangle shown is right-angled:



Call the radius of the big circle r. By Pythagoras,

$$\begin{aligned} &(r+1)^2 + 3^2 = (r+2)^2 \\ \Longrightarrow\ &r = 3. \end{aligned}$$

4609. (a) If $\ddot{x} = -\omega^2 x$, then, multiplying by the mass, $m\ddot{x} = -m\omega^2 x$, which, by NII, is

$$F = -m\omega^2 x.$$

Since $-m\omega^2$ is constant and negative, this is direct proportionality $F = kx$ with $k < 0$: force acts back towards the origin.

(b) i. Differentiating,

$$\begin{aligned} &x = A\sin(\omega t + \varepsilon) \\ \Longrightarrow\ &\dot{x} = A\omega\cos(\omega t + \varepsilon) \\ \Longrightarrow\ &\ddot{x} = -A\omega^2\sin(\omega t + \varepsilon) \\ &\quad = -\omega^2 x, \text{ as required.} \end{aligned}$$

ii. The range is $[-4, 4]$, so amplitude $A = 4$. Substituting $t = 0$ and $x = -2$,

$$\begin{aligned} &-2 = 4\sin\varepsilon \\ \Longrightarrow\ &\sin\varepsilon = -\tfrac{1}{2} \\ \therefore\ &\varepsilon = \tfrac{7\pi}{6}, \tfrac{11\pi}{6}. \end{aligned}$$

4610. We use the tabular integration method for parts. Let I be the indefinite integral of $e^{2x}\sin 4x$.

Signs	Derivatives	Integrals
$+$	$\sin 4x$	e^{2x}
$-$	$4\cos 4x$	$\frac{1}{2}e^{2x}$
$+$	$-16\sin 4x$	$\frac{1}{4}e^{2x}$

The bottom line (as a product) is -4 times the top line. Hence, the original integral I can be written in terms of I. The parts formula gives

$$\begin{aligned} &I = \tfrac{1}{2}e^{2x}\sin 4x - e^{2x}\cos 4x - 4I \\ \Longrightarrow\ &5I = e^{2x}\left(\tfrac{1}{2}\sin 4x - \cos 4x\right) \\ \Longrightarrow\ &I = \tfrac{1}{5}e^{2x}\left(\tfrac{1}{2}\sin 4x - \cos 4x\right). \end{aligned}$$

So, the definite integral is

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} e^{2x}\sin 4x\,dx\\ &= \tfrac{1}{5}\Big[e^{2x}\big(\tfrac{1}{2}\sin 4x-\cos 4x\big)\Big]_0^{\frac{\pi}{2}}\\ &= \tfrac{1}{5}\big(-e^{\pi}-(-1)\big)\\ &= \tfrac{1}{5}\big(1-e^{\pi}\big), \text{ as required.}\end{aligned}$$

— Nota Bene —

To understand the tabular integration method, it is instructive to perform parts twice manually and compare. In a cyclic example such as $e^{2x}\sin 4x$, the integral left at the end is read off along a row of the table.

This doesn't happen for a non-cyclic example: when integrating x^3e^x, all integrals are *performed* by the table (as opposed to being turned into new integrals), so no entry is read off a single line.

4611. (a) The equation of the ellipse is

$$\left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2=1.$$

The first Pythagorean trig identity allows us to express $\frac{x}{3}=\cos\theta$ and $\frac{y}{4}=\sin\theta$, giving $x=3\cos\theta$ and $y=4\sin\theta$.

(b) The gradient is given by

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{dt}\div\frac{dx}{dt}\\ &=\frac{4\cos\theta}{-3\sin\theta}\\ &\equiv -\tfrac{4}{3}\cot\theta.\end{aligned}$$

So, the gradient of the normal is $m=\frac{3}{4}\tan\theta$. The equation of the normal is

$$y-4\sin\theta=\tfrac{3}{4}\tan\theta(x-3\cos\theta).$$

Substituting $y=0$ for the x intercept,

$$\begin{aligned}&-4\sin\theta=\tfrac{3}{4}\tan\theta(x-3\cos\theta)\\ \Longrightarrow\ &-16\sin\theta=3x\tan\theta-9\sin\theta\\ \Longrightarrow\ &-7\sin\theta=3x\tan\theta\\ \Longrightarrow\ &-7\cos\theta=3x\\ \Longrightarrow\ &x=-\tfrac{7}{3}\cos\theta, \text{ as required.}\end{aligned}$$

4612. Using a polynomial solver, the roots of the quartic are $x=\pm\frac{1}{3},\frac{7}{2},-\frac{1}{4}$. So, it must factorise as

$$k(3x-1)(3x+1)(2x-7)(4x+1).$$

Considering the constant term, $k=2$. So, the full factorisation is

$$2(3x-1)(3x+1)(2x-7)(4x+1).$$

4613. (a) Setting the derivative to zero,

$$\begin{aligned}&7x^6-56x^3+49=0\\ \Longrightarrow\ &7(x^3-1)(x^3-7)=0\\ \Longrightarrow\ &x=1,\sqrt[3]{7}.\end{aligned}$$

The second derivative is $42x^5-168x^2$.

$$\begin{aligned}42x^5-168x^2\,\big|_{x=1}&=-126<0\\ 42x^5-168x^2\,\big|_{x=\sqrt[3]{7}}&=126\times 7^{\frac{2}{3}}>0.\end{aligned}$$

So, the curve has a maximum at $x=1$ and a minimum at $x=\sqrt[3]{7}$, and has no other SPs.

(b) Solving for x intercepts,

$$\begin{aligned}&x^7-14x^4+49x=0\\ \Longrightarrow\ &x(x^6-14x^3+49)=0\\ \Longrightarrow\ &x(x^3-7)^2=0.\end{aligned}$$

This has a single root at $x=0$, which gives a sign change at the origin. The other factor is squared, so it is always non-negative, and can't produce a sign change. Hence, the curve has exactly one sign change, as required.

4614. (a) The integral gives the area of the quarter circle in the positive quadrant, whose x limits are 0 and r.

(b) Let $x=r\sin\theta$, so that $dx=r\cos\theta\,d\theta$. The new limits are $\theta=0$ to $\theta=\frac{\pi}{2}$. This gives

$$\begin{aligned}&\int_0^r\sqrt{r^2-x^2}\,dx\\ \equiv&\int_0^{\frac{\pi}{2}}\sqrt{r^2-r^2\sin^2\theta}\cdot r\cos\theta\,d\theta\\ \equiv&\int_0^{\frac{\pi}{2}}\sqrt{r^2\cos^2\theta}\cdot r\cos\theta\,d\theta\\ \equiv&\ r^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta.\end{aligned}$$

Using a double-angle formula,

$$\begin{aligned}A&=4r^2\int_0^{\frac{\pi}{2}}\tfrac{1}{2}(\cos 2\theta+1)\,d\theta\\ &\equiv 2r^2\Big[-\tfrac{1}{2}\sin 2\theta+\theta\Big]_0^{\frac{\pi}{2}}\\ &\equiv 2r^2\big(\tfrac{\pi}{2}-0\big)\\ &\equiv \pi r^2, \text{ as required.}\end{aligned}$$

4615. Since X has a binomial distribution, it takes only discrete values. So, the required probability is

$$p=\mathbb{P}\left(X\in\{2,3,4\}\mid X\in\{3,4,5\}\right).$$

Using the conditional probability formula, this is

$$\begin{aligned}p&=\frac{\mathbb{P}\left(X\in\{3,4\}\right)}{\mathbb{P}\left(X\in\{3,4,5\}\right)}\\ &=\frac{0.16648+0.23837}{0.16648+0.23837+0.234033}\\ &=0.634\text{ (3sf).}\end{aligned}$$

4616. (a) With the given definitions, $b = a^2$ is

$$\begin{aligned}\tfrac{1}{\sqrt{2}}(x+y) &= \left(\tfrac{1}{\sqrt{2}}(x-y)\right)^2 \\ \implies x+y &= \tfrac{1}{\sqrt{2}}(x-y)^2.\end{aligned}$$

(b) Substituting $x = \sqrt{2}$, $y = 0$, we get $a = 1$. So, by symmetry, the values are $a = \pm 1$.

(c) The area of the shaded region is given by the area of the rectangle minus the area between $b = a^2$ and $b = 0$. The area of the rectangle is 2. The limits of the integral are $a = \pm 1$. The shaded area is given, therefore, by

$$A = 2 - \int_{a=-1}^{a=1} a^2 \, da.$$

(d) Evaluating the above,

$$A = 2 - \left[\tfrac{1}{3}a^3\right]_{-1}^{1} = \tfrac{4}{3}.$$

4617. This is a separable differential equation:

$$\begin{aligned} 2x\cot y &= \frac{dy}{dx}(x^2+1) \\ \implies \int \frac{2x}{x^2+1}\,dx &= \int \tan y \, dy \\ \implies \ln(x^2+1) &= \ln|\sec y| + c \\ \therefore\quad x^2+1 &= A\sec y \\ \implies (x^2+1)\cos y &= A. \end{aligned}$$

Substituting $x = y = 0$ gives $A = 1$. So, the path of the particle may be expressed as

$$(x^2+1)\cos y = 1, \text{ as required.}$$

4618. The relevant derivative is

$$\frac{dx}{dt} = \cos 2t + \cos t.$$

The t values (since both x and y are periodic with period 2π) are $t_1 = 0$ and $t_2 = 2\pi$. Using the parametric integration formula, the area is

$$\begin{aligned} &\int_0^{2\pi} \left(\tfrac{1}{2}\cos 2t + \cos t\right)\left(\cos 2t + \cos t\right) dt \\ = &\int_0^{2\pi} \tfrac{1}{2}\cos^2 2t + \cos t \cos 2t + \cos^2 t \, dt \end{aligned}$$

Notate this $I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_0^{2\pi} \tfrac{1}{2}\cos^2 2t + \cos^2 t \, dt, \\ I_2 &= \int_0^{2\pi} \cos t \cos 2t \, dt. \end{aligned}$$

We calculate these as follows:

① Using $\cos^2 t \equiv \tfrac{1}{2}(\cos 2t + 1)$,

$$\begin{aligned} I_1 &= \int_0^{2\pi} \tfrac{1}{2}\cos^2 2t + \cos^2 t \, dt \\ &= \int_0^{2\pi} \tfrac{1}{4}(\cos 4t + 1) + \tfrac{1}{2}(\cos 2t + 1)\, dt \\ &= \int_0^{2\pi} \tfrac{1}{4}\cos 4t + \tfrac{1}{2}\cos 2t + \tfrac{3}{4}\, dt \\ &= \left[\tfrac{1}{16}\sin 4t + \tfrac{1}{4}\sin 2t + \tfrac{3}{4}t\right]_0^{2\pi} \\ &= \tfrac{3\pi}{2}. \end{aligned}$$

② Using $\cos 2t \equiv 1 - 2\sin^2 t$,

$$\begin{aligned} I_2 &= \int_0^{2\pi} \cos t(1 - 2\sin^2 t)\, dt \\ &= \int_0^{2\pi} \cos t - 2\cos t \sin^2 t \, dt. \end{aligned}$$

Integrating the second term by inspection,

$$I_2 = \left[\sin t - \tfrac{2}{3}\sin^3 t\right]_0^{2\pi} = 0.$$

So, the area is $I_1 + I_2 = \frac{3\pi}{2}$, as required.

4619. The ingredients for the sketch are as follows:

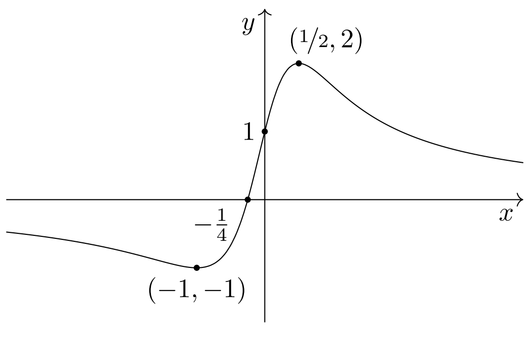
① The axis intercepts are at $(-1/4, 0)$ and $(0, 1)$.

② The denominator has no real roots, so the curve has no vertical asymptotes. So, there are no discontinuities over $\mathbb{R}$.

③ For SPs, $4(2x^2+1) - (4x+1)(4x) = 0$
$\implies x = -1, 1/2.$
So, there are SPs at $(-1, -1)$ and $(1/2, 2)$.

④ As $x \to \infty$, $y \to 0^+$. As $x \to -\infty$, $y \to 0^-$. The x axis is a horizontal asymptote.

The above is enough to guarantee the behaviour, so there is no need to classify the SPs with the second derivative. The graph is



4620. Given assignations of the function f, we find the probabilities as follows:

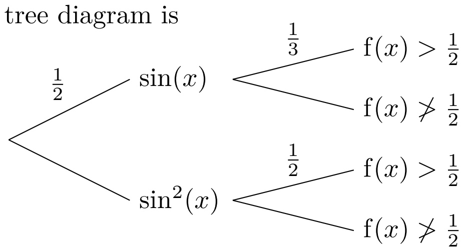
① $\sin x > \frac{1}{2} \implies x \in (\pi/6, 5\pi/6)$. This gives

$$\mathbb{P}\left(\sin(x) > \tfrac{1}{2}\right) = \tfrac{1}{3}.$$

② We know that $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$. Since this is centred on $\frac{1}{2}$, symmetry dictates that, with f as $\sin^2$, the probability is

$$\mathbb{P}\left(\sin^2 x > \tfrac{1}{2}\right) = \tfrac{1}{2}.$$

The tree diagram is



This gives

$$\mathbb{P}\big(\sin \mid \mathrm{f}(x) > \tfrac{1}{2}\big) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{2}} = \frac{2}{5}.$$

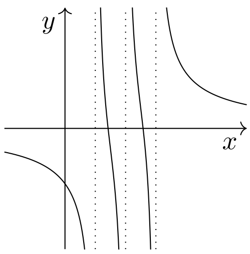
4621. (a) Written longhand, the graph is

$$y = \frac{1}{x-1} + \frac{1}{x-2} + ... + \frac{1}{x-n}.$$

This has a set of n vertical asymptotes at $x = 1, 2, ..., n$. There is also a horizontal asymptote at $y = 0$, as every fraction tends to zero for large n. Hence, the graph has $n+1$ asymptotes in total.

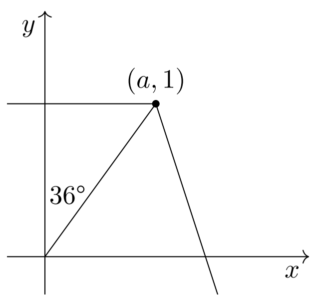
(b) Consider the terms separately. Each graph $y = {}^1\!/_{x-r}$ is a translated version of $y = {}^1\!/_x$. This is decreasing everywhere it is defined. Hence, since each term is decreasing, the entire graph is decreasing.

Also, the asymptotes are single asymptotes, meaning that there is a sign change at each. For large negative x, we have $y < 0$, and for large positive x, we have $y > 0$. So, with e.g. $n = 3$, the graph is



Since the graph is decreasing everywhere, there are no roots outside the vertical asymptotes. Furthermore, there must be one between each successive pair of vertical asymptotes. So, there are $n-1$ roots, as required.

4622. (a) E_1 is the uppermost edge. Its endpoints are at $(-a, 1)$ and $(a, 1)$. The angle subtended at the origin by these is $\frac{360}{5} = 72°$. Considering only the positive quadrant, we have a right-angled triangle with vertices at O, $(0, 1)$ and $(a, 1)$.



(b) 72°.

(c) The vertex in common to E_1 and E_2 has coordinates $(\tan 36°, 1)$, so position vector $(\tan 36°)\mathbf{i}+\mathbf{j}$. A direction vector running from this point along E_2 is $(\cos 72°)\mathbf{i} - (\sin 72°)\mathbf{j}$. With t as a parameter, we can express E_2 with the equation

$$\begin{aligned}\mathbf{r} &= (\tan 36°)\mathbf{i} + \mathbf{j} + t\big((\cos 18°)\mathbf{i} - (\sin 18°)\mathbf{j}\big) \\ &\equiv (\tan 36° + t\cos 18°)\mathbf{i} + (1 - t\sin 18°)\mathbf{j}.\end{aligned}$$

(d) At $t = 0$, the above equation gives $(\tan 36°, 1)$, which is the vertex between E_1 and E_2. By Pythagoras, the length of the direction vector is 1. Edge E_2 must have the same length as E_1, which is $2\tan 36°$. So, $t \in [0, 2\tan 36°]$.

4623. The second derivative must be zero at $x = 0$ and $x = 4$, where there are points of inflection. So, it must take the form

$$\begin{aligned}\frac{d^2y}{dx^2} &= kx(x-4) \\ &\equiv kx^2 - 4kx.\end{aligned}$$

Integrating this twice, $y = \frac{1}{12}kx^4 + ...$, so $k = 12$. The first derivative is

$$\frac{dy}{dx} = 4x^3 - 24x^2 + c.$$

There is a stationary point at $x = 3$, so

$$\begin{aligned}0 &= 4\cdot 3^3 - 24\cdot 3^2 + c \\ \Longrightarrow c &= 108.\end{aligned}$$

Integrating again,

$$y = x^4 - 8x^3 + 108x + d.$$

Substituting $(3, -11)$ gives $d = -200$. Hence, the equation of the quartic is

$$y = x^4 - 8x^3 + 108x - 200.$$

4624. (a) Let $u = \ln x$, so that $x = e^u$. This gives $dx = u\,du$. The upper limit $x = 1$ gives $u = 0$. We cannot evaluate u at $x = 0$, however. So, we set up a limit. Let the lower limit be $x = \delta$, where $\delta \to 0$. Then the lower limit is $u = \ln\delta$. Let $k = \ln\delta$. As $\delta \to 0$, $k \to -\infty$. So,

$$I = \lim_{k\to-\infty} \int_k^0 u^2 e^u\,du.$$

(b) We integrate twice by parts, using the tabular integration method:

Signs	Derivatives	Integrals
+	u^2	e^u
−	$2u$	e^u
+	2	e^u
−	0	e^u

This gives $(u^2-2u+2)e^u$. So,

$$\begin{aligned} I &= \lim_{k\to-\infty} \Big[(u^2-2u+2)e^u\Big]_k^0 \\ &= \lim_{k\to-\infty} \big(2-(k^2-2k+2)e^k\big). \end{aligned}$$

In the right-hand term, the exponential, which tends to 0, dominates the polynomial, which tends to infinity. So, the right-hand term tends to zero, giving $I=2$.

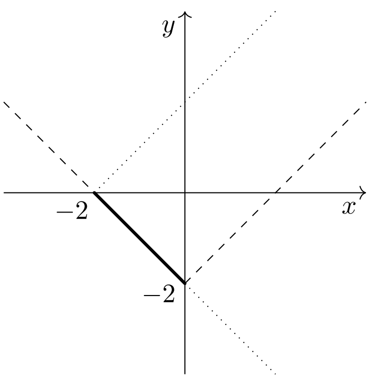
4625. (a) $\cos\theta \approx 1-\frac{1}{2}\theta^2$.

(b) Let $\cos\theta \approx 1-\frac{1}{2}\theta^2+k\theta^3$. We know that the function and its first two derivatives match at $\theta=0$. Comparing third derivatives, we need $-\sin\theta = 6k$ at $\theta=0$. This gives $k=0$.

(c) Let $\cos\theta \approx 1-\frac{1}{2}\theta^2+k\theta^4$. Comparing fourth derivatives, we need $\cos\theta=24k$ at $\theta=0$. This gives $k=\frac{1}{24}$. The approximation is

$$\cos\theta \approx 1-\tfrac{1}{2}\theta^2+\tfrac{1}{24}\theta^4.$$

4626. Graphically, the equations are as shown:



The solid line represents the points which satisfy both equations. Algebraically, this set is

$$\Big\{(x,y)\in\mathbb{R}^2 : -2\le x\le 0, y=-2-x\Big\}.$$

——— Nota Bene ———

There is no unique way to represent this kind of set. Another answer would be

$$\Big\{(x,y)\in\mathbb{R}^2 : x=-1+t, y=-1-t, t\in[-1,1]\Big\}.$$

4627. Using the approximations given,

$$\begin{aligned} \tan x &\equiv \frac{\sin x}{\cos x} \\ &\approx \big(x-\tfrac{1}{6}x^3\big)\big(1-\tfrac{1}{2}x^2\big)^{-1}. \end{aligned}$$

Expanding binomially up to the term in x^2

$$\big(1-\tfrac{1}{2}x^2\big)^{-1} \approx 1+\tfrac{1}{2}x^2.$$

Neglecting terms in x^4 and above, this gives

$$\begin{aligned} \tan x &\approx \big(x-\tfrac{1}{6}x^3\big)\big(1+\tfrac{1}{2}x^2\big) \\ &\equiv x+\tfrac{1}{2}x^3-\tfrac{1}{6}x^3+... \\ &\approx x+\tfrac{1}{3}x^3, \text{ as required.} \end{aligned}$$

4628. Differentiating the first equation with respect to x,

$$1+\frac{dy}{dx}=\frac{dz}{dx}.$$

Substituting the second and third into this,

$$\begin{aligned} 1+2y &= 3y^2 \\ \Longrightarrow y &= 1, -\tfrac{1}{3}. \end{aligned}$$

Since y is non-negative, $y=1$. This means that $\frac{dy}{dx}=2$ and therefore

$$y=2x+c.$$

Both c and y are constant, so y is also constant. Hence, so is x. But this means that the given derivatives are not well defined: the rate of change with respect to a constant has no meaning.

So, the model is not well defined.

4629. The curve is symmetrical in $x=0$, so we need only consider the positive quadrant, doubling the result. Setting $x=0$ gives $y=0,1$. So, the area is given by

$$\begin{aligned} A &= 2\int_0^1 \sqrt{y^2-y^3}\,dy \\ &= 2\int_0^1 \sqrt{y^2(1-y)}\,dy \\ &= 2\int_0^1 y\sqrt{1-y}\,dy. \end{aligned}$$

Let $u=1-y$, so that $du=-dy$. The new limits are $u=1$ to $u=0$. Enacting the substitution,

$$\begin{aligned} A &= 2\int_1^0 (1-u)\sqrt{u}\cdot -du \\ &= 2\int_0^1 u^{\frac{1}{2}}-u^{\frac{3}{2}}\,du \\ &= 2\Big[\tfrac{2}{3}u^{\frac{3}{2}}-\tfrac{2}{5}u^{\frac{5}{2}}\Big]_0^1 \\ &= \tfrac{8}{15}. \end{aligned}$$

4630. Consider the following cases:

① If $k<0$, then the first factor provides two roots $x=\pm\sqrt{-k}$. This cannot be the case.

② If $k=0$, then the second factor provides two roots $x=0,-4$ This cannot be the case.

③ If $k>0$, the first factor has no real roots. The second factor factor must therefore have precisely one. So, $\Delta=16-4k=0$.

The only option is $k=4$.

4631. (a) The derivative is

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}.$$

At $x = a - 1$, therefore, the gradient is

$$m = -\frac{1}{a^2}.$$

So, the equation of the tangent at A is

$$\begin{aligned} y - \frac{1}{a} &= -\frac{1}{a^2}\big((x - (a-1)\big) \\ \implies y &= -\frac{1}{a^2}x + \frac{a-1}{a^2} + \frac{a}{a^2} \\ &\equiv -\frac{1}{a^2}x + \frac{2a-1}{a^2}, \text{ as required.} \end{aligned}$$

(b) Substituting $(0, 0)$ into the above,

$$\begin{aligned} 0 &= \frac{2a-1}{a^2} \\ \implies a &= \tfrac{1}{2}. \end{aligned}$$

Substituting this back in, the equation of the tangent which passes through O is $y = -4x$.

4632. (a) The sub-interval has length $0.629 - 0.231$, so 0.398. Hence, the probability that a random number lies in the sub-interval should be 0.398. With $B(100, 0.398)$, the expected number is $np = 39.8$.

(b) Let p be the probability that a random number lies in $[0.231, 0.629]$. The hypotheses are

$$\begin{aligned} H_0 &: p = 0.398, \\ H_1 &: p \neq 0.398. \end{aligned}$$

This is a two-tailed test. For the upper tail,

$$\begin{aligned} \mathbb{P}(X \geq 53) &= 0.0051 > 0.5\%, \\ \mathbb{P}(X \geq 54) &= 0.0028 < 0.5\%. \end{aligned}$$

So, the upper critical region is $\{54, 55, ..., 100\}$. In the sample, 55 numbers lay in the sub-interval. This is in the critical region. So, there is sufficient evidence, at the 1% level, to reject H_0. The random number generator seems not to be working correctly.

4633. The equation of the curve is a quadratic in y. In the quadratic formula, the positive root is

$$y = \frac{-x^2 + \sqrt{x^4 + 8x}}{2x}.$$

So, the area of the shaded region is given by

$$A = \int_1^3 \frac{-x^2 + \sqrt{x^4 + 8x}}{2x}\, dx.$$

Evaluating with the numerical integration facility on a calculator, $A = 0.9480$ (4sf).

4634. The equations factorise as follows:

$$\begin{aligned} (-3 + x)(2 + y) &= 0, \\ (x^2 + 9)(y^2 - 4) &= 0. \end{aligned}$$

The first equation is satisfied iff either $x = 3$ or $y = -2$. The second equation, since the factor $(x^2 + 9)$ has no roots, is satisfied iff $y = \pm 2$.

- If $y = 2$, then $x = 3$.
- If $y = -2$, then the first equation is satisfied automatically, and x can be anything.

The solution set is $\{(2, 3)\} \cup \big\{(x, -2) \in \mathbb{R}^2\big\}$.

4635. (a) The mass of chain on each slope is proportional to the length on that slope. Without loss of generality, let the constant of proportionality be 1, so the total mass is 2. Send the chain rightwards, calling the mass on the right-hand slope $1 + x$. The mass on the left-hand slope is $1 - x$.

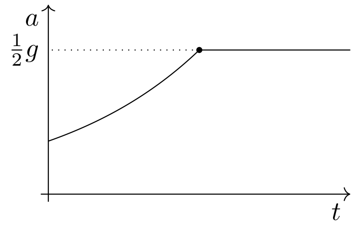
Resolving along the chain, all internal tensions cancel; only the components of weight remain. These are $(1+x)g \sin 30°$ and $(1-x)g \sin 30°$ Newtons. So, NII for the whole chain is

$$\begin{aligned} \tfrac{1}{2}(1+x)g - \tfrac{1}{2}(1+x)g &= 2\frac{d^2x}{dt^2} \\ \implies \frac{d^2x}{dt^2} &= \tfrac{1}{2}gx. \end{aligned}$$

This model breaks down when there is no more chain on the left-hand slope, which is at $x = 1$. Beyond that point, the chain can be modelled as a particle, and the acceleration is $\frac{1}{2}g$:

$$\frac{d^2x}{dt^2} = \begin{cases} \frac{1}{2}gx & 0 \leq x < 1, \\ \frac{1}{2}g & x \geq 1. \end{cases}$$

(b) At first, the acceleration grows exponentially. It is then constant for $x \geq 1$. It is continuous at the change in behaviour:



(c) For $0 \leq x < 1$, $\ddot{x} = \frac{1}{2}gx$. This is a second-order DE. Differentiating the proposed solution,

$$\begin{aligned} x &= \tfrac{u}{k}e^{kt} \\ \implies \dot{x} &= ue^{kt}. \end{aligned}$$

Substituting $t = 0$ gives $\dot{x} = u$, which is the correct value. Differentiating again,

$$\begin{aligned} \ddot{x} &= uke^{kt} \\ &= k^2 x. \end{aligned}$$

So, with $k^2 = \frac{1}{2}g$, the DE is satisfied. Taking the positive square root, $k = \sqrt{g/2}$.

4636. Differentiating the area with respect to t:

$$\begin{aligned} A &= \pi r^2 \\ \Longrightarrow \dot{A} &= 2\pi r \times \dot{r} \\ &= \frac{2\pi}{r}. \end{aligned}$$

Substituting into the LHS of the proposed equation,

$$\begin{aligned} \dot{A} \times \sqrt{A} &= \frac{2\pi}{r} \times \sqrt{\pi} r \\ &= 2\pi^{\frac{3}{2}}. \end{aligned}$$

So, the required relationship holds: $k = 2\pi^{\frac{3}{2}}$.

4637. Let the position vectors of vertices A, B, C be $\mathbf{a}, \mathbf{b}, \mathbf{c}$, relative to some origin. The midpoint M of BC then has position vector

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c}).$$

Consider the position vector $\mathbf{p}$ of the point P which divides AM in the ratio $2 : 1$. This is

$$\begin{aligned} \mathbf{p} &= \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{m} \\ &\equiv \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\big(\tfrac{1}{2}(\mathbf{b} + \mathbf{c})\big) \\ &\equiv \tfrac{1}{3}\mathbf{a} + \tfrac{1}{3}\mathbf{b} + \tfrac{1}{3}\mathbf{c}. \end{aligned}$$

Since this result is symmetrical in $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the same point P must also lie on the other two medians. Hence, the medians are concurrent. □

4638. The relevant result here is

$$\int a^x \, dx = \frac{1}{\ln a} a^x + c.$$

Using this,

$$\begin{aligned} &\int_0^1 4^x - 2^x \, dx \\ &= \left[\frac{1}{\ln 4} 4^x - \frac{1}{\ln 2} 2^x\right]_0^1 \\ &= \left(\frac{4}{\ln 4} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 4} - \frac{1}{\ln 2}\right). \end{aligned}$$

Simplifying and using $\ln 4 = 2 \ln 2$, this is

$$\begin{aligned} &\frac{3}{\ln 4} - \frac{1}{\ln 2} \\ &= \frac{3}{\ln 4} - \frac{2}{\ln 4} \\ &= \frac{1}{\ln 4}. \end{aligned}$$

This can also be written as $\log_4 e$.

——— Nota Bene ———

The quoted result can be proved by writing $a^x \equiv e^{x \ln a}$ and using the reverse chain rule.

4639. (a) Simplifying and writing in column vectors,

$$\mathbf{r} = r \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}.$$

Differentiating with respect to time,

$$\mathbf{v} = r\omega \begin{pmatrix} -\sin \omega t \\ \cos \omega t \end{pmatrix}.$$

The gradients of the column vectors are $\tan t$ and $-\cot t$, which are negative reciprocals. So, $\mathbf{v}$ is perpendicular to $\mathbf{r}$.

(b) By the first Pythagorean trig identity, the magnitude of the column vector in $\mathbf{v}$ is 1. So, taking the magnitude of the velocity, speed is given by $v = r\omega$.

(c) Differentiating again,

$$\mathbf{a} = r\omega^2 \begin{pmatrix} -\cos \omega t \\ -\sin \omega t \end{pmatrix}.$$

(d) In terms of $\mathbf{r}$, the above is $\mathbf{a} = -\omega^2 \mathbf{r}$. This tells us that the acceleration is towards the centre, i.e. back along the radius/position vector. The magnitude of the acceleration is then $a = \omega^2 r$. We substitute $\omega = {}^{v}/_{r}$ in, giving the required centripetal acceleration:

$$a = \frac{v^2}{r}.$$

4640. Using double-angle formulae, the integrand is

$$\begin{aligned} &\sin^2 x \cos^2 x \\ &\equiv \tfrac{1}{4} \sin^2 2x \\ &\equiv \tfrac{1}{8}(1 - \cos 4x). \end{aligned}$$

So, the integral is

$$\begin{aligned} &\int_{-\pi}^{\pi} \sin^2 x \cos^2 x \, dx \\ &= \int_{-\pi}^{\pi} \tfrac{1}{8}(1 - \cos 4x) \, dx \\ &= \tfrac{1}{8}\left[x - \tfrac{1}{4} \sin 4x\right]_{-\pi}^{\pi} \\ &= \tfrac{1}{8}\big(\pi - (-\pi)\big) \\ &= \tfrac{1}{4}\pi, \text{ as required.} \end{aligned}$$

4641. (a) This is false: $x^4 + 1$ is a counterexample. A cubic factor requires a linear factor, and, since $x^4 + 1$ has no real roots, it can have no linear factors.

(b) This is true: every polynomial equation of odd degree has a real root. Take out the relevant factor, and we are left with a quartic factor.

(c) This is false: $x^6 + 1$ is a counterexample, with the same logic as in (a).

4642. The equation of each circle takes the form

$$(x-a)^2 + (y-r)^2 = r^2.$$

The point $(0,1)$ gives

$$\begin{aligned} a^2 + (1-r)^2 &= r^2 \\ \Longrightarrow a^2 &= 2r - 1. \end{aligned}$$

The point $(3,10)$ gives

$$\begin{aligned} (3-a)^2 + (10-r)^2 &= r^2 \\ \Longrightarrow 109 - 6a - 20r + a^2 &= 0. \end{aligned}$$

Substituting the former into the latter,

$$\begin{aligned} 109 - 6a - 20r + 2r - 1 &= 0 \\ \Longrightarrow a &= 18 - 3r \end{aligned}$$

Substituting this back in,

$$\begin{aligned} (18-3r)^2 &= 2r - 1 \\ \Longrightarrow r &= 5, \tfrac{65}{9}. \end{aligned}$$

4643. Writing $\cos 4\theta \equiv \cos(2 \cdot 2\theta)$, we use a double-angle formula twice:

$$\begin{aligned} \cos(2\cdot 2\theta) &= 2\cos^2 2\theta - 1 \\ &\equiv 2(2\cos^2\theta - 1)^2 - 1 \\ &\equiv 8\cos^2\theta - 8\cos\theta + 1. \end{aligned}$$

So, the RHS of the original identity is

$$\begin{aligned} &\frac{3 + 4\cos 2\theta + \cos 4\theta}{8} \\ \equiv\ &\frac{3 + 4(2\cos^2\theta - 1) + 8\cos^2\theta - 8\cos\theta + 1}{8} \\ \equiv\ &\frac{3 + 8\cos^2\theta - 4 + 8\cos^4\theta - 8\cos^2\theta + 1}{8} \\ \equiv\ &\frac{8\cos^4\theta}{8} \\ \equiv\ &\cos^4\theta, \text{ as required.} \end{aligned}$$

4644. (a) i. The numerator has a squared factor, so that the curve touches the x axis at $x = 0$, but doesn't cross it. Close to the root $x = 0$, the line $x = 0$ is an approximate line of symmetry.

ii. The denominator has a squared factor, meaning that the curve is undefined at $x = 9/2$. However, there is no sign change across $x = 9/2$. Close to the asymptote $x = 9/2$, the line $x = 9/2$ is an approximate line of symmetry.

(b) Rewriting algebraically as a proper fraction,

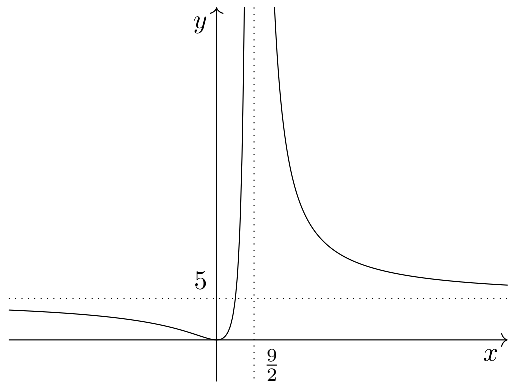
$$\frac{20x^2}{(2x-9)^2} \equiv 5 + \frac{180x - 405}{(2x-9)^2}.$$

As $x \to \pm\infty$, since the denominator is positive,

$$\frac{180x - 405}{(2x-9)^2} \to 0^{\pm}.$$

Hence, $y \to 5^{\pm}$.

(c) Putting the above together, $y = \mathrm{f}(x)$ is



4645. Separating the variables,

$$\int \frac{1}{y^2 - y}\, dx = \int \cos x \, dx.$$

Writing the y integrand in partial fractions,

$$\frac{1}{y^2-y} \equiv \frac{1}{y-1} - \frac{1}{y}.$$

We can now integrate:

$$\begin{aligned} \int \frac{1}{y-1} - \frac{1}{y}\, dx &= \int \cos x \, dx \\ \Longrightarrow \ln\left|\frac{y-1}{y}\right| &= \sin x + c \\ \therefore \quad \frac{y-1}{y} &= Ae^{\sin x}. \end{aligned}$$

The boundary conditions give $A = -1$. So,

$$\begin{aligned} \frac{y-1}{y} &= -e^{\sin x} \\ \Longrightarrow 1 - \frac{1}{y} &= -e^{\sin x} \\ \Longrightarrow \frac{1}{y} &= 1 + e^{\sin x} \\ \Longrightarrow y &= \frac{1}{1 + e^{\sin x}}. \end{aligned}$$

4646. Differentiating implicitly,

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 2\frac{dy}{dx} + 4.$$

The normal gradient is $-\frac{4}{3}$, so $\frac{dy}{dx} = \frac{3}{4}$:

$$\begin{aligned} \tfrac{9}{4}y^2 + y + \tfrac{3}{4}x &= \tfrac{11}{2} \\ \Longrightarrow x &= -3y^2 - \tfrac{4}{3}y + \tfrac{22}{3}. \end{aligned}$$

Substituting this into the equation of the curve,

$$\begin{aligned} &y^3 + \left(-3y^2 - \tfrac{4}{3}y + \tfrac{22}{3}\right)y \\ &\quad = 2y + 4\left(-3y^2 - \tfrac{4}{3}y + \tfrac{22}{3}\right) - 10 \\ \Longrightarrow\ &3y^3 - 16y^2 - 16y + 29 = 0 \\ \Longrightarrow\ &y = 1, \tfrac{13 \pm \sqrt{517}}{6}. \end{aligned}$$

The point in question has integer coordinates, so $y = 1$. Substituting into $x = -3y^2 - \frac{4}{3}y + \frac{22}{3}$ gives $x = 3$. Putting the point $(3,1)$ into $4x + 3y = k$, we get $k = 15$.

4647. The intersection between the curves is at

$$x\ln(x^2+1)-\sqrt{4-x}=0.$$

The N-R iteration is $x_{n+1}=x_n-\mathrm{g}(x_n)$, where

$$\mathrm{g}: x\mapsto \frac{x\ln(x^2+1)-\sqrt{4-x}}{\ln(x^2+1)+\frac{2x^2}{x^2+1}+\frac{1}{2}(4-x)^{-\frac{1}{2}}}.$$

Running this with $x_0=1$ gives $x_1=1.52422$ and $x_n\to 1.433919$. Using the numerical integration facility on a calculator, the area is

$$\begin{aligned}A&=\int_0^{1.43} x\ln(x^2+1)\,dx+\int_{1.43}^{4}\sqrt{4-x}\,dx\\&=0.67901+2.74040\\&=3.42\text{ (3sf)}.\end{aligned}$$

4648. None of the symbols apply. Counterexamples to the two possible directions of implication are:

$\Longrightarrow$ $\mathrm{f}(x)=x^2+x$, $\mathrm{g}(x)=x$, with $a=0$. The difference $\mathrm{f}(x)-\mathrm{g}(x)=x^2$ has a factor of x^2, but $\mathrm{f}'(0)=1$.

$\Longleftarrow$ $\mathrm{f}(x)=2$, $\mathrm{g}(x)=1$, with $a=0$. Both of the derivatives are zero everywhere, but $\mathrm{f}(x)-\mathrm{g}(x)=1$, so has no factor of x^2.

4649. The first may be chosen without loss of generality. The probability that the second is different is then $\frac{6}{8}$. The probability that the third is different to the first two is then $\frac{3}{7}$. So, the probability of getting three marbles of different colours is

$$p=\tfrac{6}{8}\times\tfrac{3}{7}=\tfrac{9}{28}.$$

— Alternative Method —

With all nine marbles considered as different, there are ${}^9\mathrm{C}_3$ equally likely outcomes in the possibility space. Of these, the successful outcomes contain one of each colour. There are three choices for each marble, giving 3^3. So, the probability is

$$p=\frac{3^3}{{}^9\mathrm{C}_3}=\tfrac{9}{28}.$$

4650. Rewriting and differentiating,

$$y=\log_k x=\frac{\ln x}{\ln k}\implies\frac{dy}{dx}=\frac{1}{x\ln k}.$$

Setting the gradient to 1,

$$1=\frac{1}{x\ln k}\implies x=\frac{1}{\ln k}.$$

We want this point not only to have gradient 1, but also to lie on the line $y=x$. So, we require

$$\begin{aligned}\frac{1}{\ln k}&=\frac{\ln\frac{1}{\ln k}}{\ln k}\\ \Longrightarrow 1&=\ln\tfrac{1}{\ln k}\\ \Longrightarrow e&=\frac{1}{\ln k}\\ \Longrightarrow \ln k&=\frac{1}{e}.\end{aligned}$$

So, the x coordinate of the point of tangency is

$$\begin{aligned}x&=\frac{1}{\ln k}\\&=\frac{1}{1/e}\\&=e.\end{aligned}$$

The point of tangency is (e,e).

4651. Let $t+a=x$. This gives $dt=dx$. The new x limits are 0 and 5. Enacting the substitution,

$$I=\int_0^5\frac{1}{(x+1)(x+2)(x+3)}\,dx.$$

In partial fractions, the integrand is

$$\frac{1}{(x+1)(x+2)(x+3)}\equiv\frac{1/2}{x+1}-\frac{1}{x+2}+\frac{1/2}{x+3}.$$

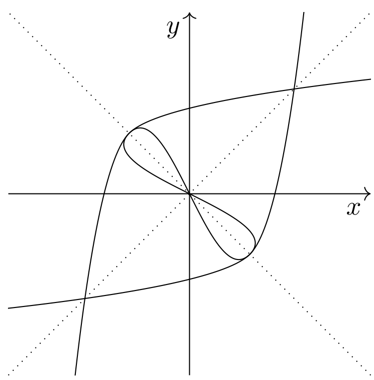
So, the integral I is

$$\begin{aligned}&\int_0^5\frac{1/2}{x+1}-\frac{1}{x+2}+\frac{1/2}{x+3}\,dx\\=&\left[\tfrac{1}{2}\ln|x+1|-\ln|x+2|+\tfrac{1}{2}\ln|x+3|\right]_0^5.\end{aligned}$$

Using log rules, this can be written as

$$\begin{aligned}&\left[\ln\frac{\sqrt{(x+1)(x+3)}}{x+2}\right]_0^5\\=&\ \ln\tfrac{\sqrt{48}}{7}-\ln\tfrac{\sqrt{3}}{2}\\=&\ \ln\tfrac{2\sqrt{48}}{7\sqrt{3}}\\=&\ \ln\tfrac{8}{7}.\end{aligned}$$

4652. The symmetry of the curves dictates that all points of intersection lie on the lines $y=\pm x$.



Solving $y=x^3-2x$ with $y=x$ gives $x=0,\pm\sqrt{3}$. Solving $y=x^3-2x$ with $y=-x$ gives $x=0,\pm 1$. So, the points of intersection are

$$(0,0),\quad(\pm\sqrt{3},\pm\sqrt{3}),\quad(\pm 1,\mp 1).$$

4653. (a) $\text{S}(1) = \frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = 1$.

(b) Starting with the LHS,

$$\begin{aligned} &\text{S}(k) + (k+1)^2 \\ \equiv\ &\tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ \equiv\ &\tfrac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big) \\ \equiv\ &\tfrac{1}{6}(k+1)\big(2k^2 + 7k + 6\big) \\ \equiv\ &\tfrac{1}{6}(k+1)(k+2)(2k+3) \\ \equiv\ &\tfrac{1}{6}(k+1)(k+2)(2(k+1)+1) \\ \equiv\ &\text{S}(k+1), \text{ as required.} \end{aligned}$$

(c) The 'sum' of the first 1 integer is 1. So, $\text{S}(1)$ gives the correct sum, as verified in part (a). Now, assume that $\text{S}(k)$ does give the sum of the first k integers. Adding $(k+1)^2$ must give the sum of the first $k+1$ integers. Hence, part (b) tells us that, if $\text{S}(n)$ is the correct formula for $n = k$, then it is correct for $n = k+1$. And, since it is correct for $n = 1$, it must therefore be correct for all n.

— Nota Bene —

This question contains the elements of a *proof by induction*, which is beyond the scope of this book. Part (a) is known as the *base case*, and part (b) is known as the *inductive step*. You can think of these by analogy with a ladder: prove that you can get onto the bottom rung of the ladder, then prove that, when on rung k, you can climb to rung $k+1$. Obviously (the formal idea is called the principle of induction), you can then climb the whole ladder!

4654. Using compound-angle formulae, the factors on the RHS have the following expansions:

$$\begin{aligned} \sin\left(\tfrac{A+B}{2}\right) &\equiv \sin\tfrac{A}{2}\cos\tfrac{B}{2} + \cos\tfrac{A}{2}\sin\tfrac{B}{2}, \\ \cos\left(\tfrac{A-B}{2}\right) &\equiv \cos\tfrac{A}{2}\cos\tfrac{B}{2} + \sin\tfrac{A}{2}\sin\tfrac{B}{2}. \end{aligned}$$

Multiplying the expansions together, the RHS of the sum-to-product formula has four terms. These, and their simplifications, are

$$\begin{aligned} 2\sin\tfrac{A}{2}\cos\tfrac{B}{2}\cos\tfrac{A}{2}\cos\tfrac{B}{2} &\equiv \sin A\cos^2\tfrac{B}{2}, \\ 2\sin\tfrac{A}{2}\cos\tfrac{B}{2}\sin\tfrac{A}{2}\sin\tfrac{B}{2} &\equiv \sin A\sin^2\tfrac{B}{2}, \\ 2\cos\tfrac{A}{2}\sin\tfrac{B}{2}\cos\tfrac{A}{2}\cos\tfrac{B}{2} &\equiv \sin B\cos^2\tfrac{B}{2}, \\ 2\cos\tfrac{A}{2}\sin\tfrac{B}{2}\sin\tfrac{A}{2}\sin\tfrac{B}{2} &\equiv \sin B\sin^2\tfrac{B}{2}. \end{aligned}$$

Using the first Pythagorean identity, the sum of the first two of these is

$$\begin{aligned} &\sin A\cos^2\tfrac{B}{2} + \sin A\sin^2\tfrac{B}{2} \\ \equiv\ &\sin A\Big(\cos^2\tfrac{B}{2} + \sin^2\tfrac{B}{2}\Big) \\ \equiv\ &\sin A. \end{aligned}$$

By the same logic, the sum of the other two is $\sin B$, giving $\sin A + \sin B$, as required.

4655. To maximise the number of pieces, every new cut must cross every existing cut. With $n = 1$, $P_1 = 2$. The second cut then crosses two existing regions, creating two new pieces. So, $P_2 = 4$. The third cut crosses three existing pieces, creating three new pieces. The iterative definition of the sequence is

$$\begin{aligned} P_1 &= 2, \\ P_n &= P_{n-1} + n. \end{aligned}$$

Since the differences increase linearly, the sequence must be quadratic. The second difference is 1, so the ordinal formula is

$$P_n = \tfrac{1}{2}n^2 + bn + c.$$

Substituting $P_1 = 2$ and $P_2 = 4$, we get

$$\begin{aligned} 2 &= \tfrac{1}{2} + b + c, \\ 4 &= 2 + 2b + c. \end{aligned}$$

Subtracting gives $b = \frac{1}{2}$, so $c = 1$. Hence,

$$P_n = \frac{n^2 + n + 2}{2}, \text{ as required.}$$

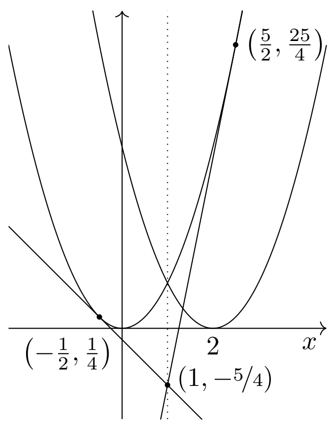
4656. The problem is symmetrical in the line $x = 1$. So, consider a generic tangent to $y = x^2$, at $x = a$. This has equation

$$\begin{aligned} y - a^2 &= 2a(x - a) \\ \Longrightarrow y &= 2ax - a^2. \end{aligned}$$

Substituting $(1, -5/4)$,

$$\begin{aligned} -\tfrac{5}{4} - a^2 &= 2a(1 - a) \\ \Longrightarrow a &= \tfrac{1}{2}, \tfrac{5}{2}. \end{aligned}$$

Sketching the scenario, the tangents to $y = x^2$ at $x = -1/2$ and $x = 5/2$ are



As seen, the tangent at $x = 5/2$ intersects the other parabola. So, the tangent in question is $x = -1/2$. This has gradient -1. The symmetrical tangent on the other parabola has gradient $+1$. These meet at right angles, as required.

4657. Expressing the summand in partial fractions,

$$\frac{4}{r^2+2r} = \frac{2}{r} - \frac{2}{r+2}.$$

Writing the sum longhand,

$$\begin{aligned}&\sum_{r=1}^{\infty} \frac{2}{r} - \frac{2}{r+2}\\ &= \left(\frac{2}{1}-\frac{2}{3}\right) + \left(\frac{2}{2}-\frac{2}{4}\right) + \left(\frac{2}{3}-\frac{2}{5}\right) + ...\end{aligned}$$

In the above, there are terms of the form $-\frac{2}{i}$ and $+\frac{2}{i}$ for all $i \geq 3$. So, after the infinite limit has been taken, all that remains is $\frac{2}{1}+\frac{2}{2}$, the first and third terms. Therefore,

$$\sum_{r=1}^{\infty} \frac{4}{r^2+2r} = 3, \text{ as required.}$$

4658. Let $\arctan x = y$, so $\tan y = x$.

$$\begin{aligned}\cos(\arctan x) &= \cos y\\ &\equiv \frac{1}{\sec y}\\ &= \frac{1}{\sqrt{\sec^2 y}}\\ &\equiv \frac{1}{\sqrt{1+\tan^2 y}}\\ &= \frac{1}{\sqrt{1+x^2}}, \text{ as required.}\end{aligned}$$

— Nota Bene —

The justification for taking the positive square root, i.e. for writing $\sec y = \sqrt{\sec^2 y}$, is that the range of the arctan function is $(-\pi/2, \pi/2)$. Over this domain, cosine is non-negative. So, since $\sec x \geq 0$, it can be expressed as the positive root of its square.

4659. The derivative of $\cos x$ is $-\sin x$. The gradient of the normal to $y = \cos x$ is duly $1/\sin x$. At $x = \theta$, this is $1/\sin\theta$. Since θ is small, we approximate this with $1/\theta$. Hence, also using the cosine small-angle approximation, the equation of the normal at $\left(\theta, 1-\frac{1}{2}\theta^2\right)$ is approximately

$$\begin{aligned}& y - 1 + \tfrac{1}{2}\theta^2 = \tfrac{1}{\theta}(x-\theta)\\ \Longrightarrow\ & y = \tfrac{1}{\theta}x + 1 - \tfrac{1}{\theta}\theta - \tfrac{1}{2}\theta^2\\ & \quad\equiv \tfrac{1}{\theta}x - \tfrac{1}{2}\theta^2.\end{aligned}$$

Setting $y = 0$ and solving gives $x = \frac{1}{2}\theta^3$. For small θ, this is negligible, as required.

4660. Consider the image of $y = x^p + x^{p-1}$ under stretches by scale factor c and d in the x and y directions. Algebraically, this is

$$\begin{aligned}y &= d\left(\frac{x}{c}\right)^p + d\left(\frac{x}{c}\right)^{p-1}\\ &\equiv \frac{d}{c^p}x^p + \frac{d}{c^{p-1}}x^{p-1}.\end{aligned}$$

So, equating coefficients,

$$a = \frac{d}{c^p}, \quad b = \frac{d}{c^{p-1}}.$$

Dividing these, $\dfrac{b}{a} = \dfrac{c^p}{c^{p-1}} = c$. This gives

$$d = ac^p = a\left(\frac{b}{a}\right)^p \equiv \frac{b^p}{a^{p-1}}.$$

Substituting these values,

$$\begin{aligned}cd &= \frac{b}{a}\cdot\frac{b^p}{a^{p-1}}\\ &\equiv \frac{b^{p+1}}{a^p}, \text{ as required.}\end{aligned}$$

4661. (a) Separating the first term,

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \sum_{n=2}^{\infty}\frac{1}{n^2}.$$

For $n \geq 2$, we know $0 < n(n-1) < n^2$, so

$$\frac{1}{n(n-1)} > \frac{1}{n^2}.$$

So, from equality, the new RHS is larger:

$$\zeta(2) < 1 + \sum_{n=2}^{\infty}\frac{1}{n(n-1)}.$$

(b) Expressing the summand in partial fractions,

$$\begin{aligned}\zeta(2) &< 1 + \sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)\\ &= 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ...\end{aligned}$$

All terms except for the first two cancel. So, in the infinite limit, $\zeta(2) < 2$. Hence, since all terms are positive, it must converge to a finite sum, as required.

4662. Let I be the integral of $2e^{-x}\sin x$. We integrate by parts, using the tabular integration method:

Signs	Derivatives	Integrals
$+$	$2\sin x$	e^{-x}
$-$	$2\cos x$	$-e^{-x}$
$+$	$-2\sin x$	e^{-x}

This is periodic. So, we can write

$$\begin{aligned}& I = -2e^{-x}\sin x - 2e^{-x}\cos x - I\\ \Longrightarrow\ & 2I = -2e^{-x}\sin x - 2e^{-x}\cos x\\ \Longrightarrow\ & I = -e^{-x}(\sin x + \cos x).\end{aligned}$$

So, area of the first shaded region is

$$\begin{aligned} A_1 &= \int_0^{\pi} 2e^{-x}\sin x\,dx \\ &= \Big[-e^{-x}(\sin x+\cos x)\Big]_0^{\pi} \\ &= e^{-\pi}+1. \end{aligned}$$

The area of the second shaded region is

$$\begin{aligned} A_2 &= -\int_{\pi}^{2\pi} 2e^{-x}\sin x\,dx \\ &= \Big[-e^{-x}(\sin x+\cos x)\Big]_{\pi}^{2\pi} \\ &= e^{-2\pi}+e^{-\pi} \\ &= e^{-\pi}(e^{-\pi}+1) \\ &= e^{-\pi}A_1. \end{aligned}$$

This pattern continues, with the areas A_i forming a GP. The first term is $1+e^{-\pi}$ and the common ratio is $e^{-\pi}$. So, the infinite sum is

$$\begin{aligned} A &= \frac{a}{1-r} \\ &= \frac{1+e^{-\pi}}{1-e^{-\pi}} \\ &= \frac{e^{\pi}+1}{e^{\pi}-1}, \text{ as required.} \end{aligned}$$

4663. Setting the discriminants to zero,

$$\begin{aligned} \Delta_1 &= 4p^2-4q=0, \\ \Delta_2 &= q^2-4(p^2-1)=0. \end{aligned}$$

The former is $q=p^2$. Substituting into the latter,

$$\begin{aligned} & p^4-4p^2+4=0 \\ \Longrightarrow\ & (p^2-2)^2=0 \\ \Longrightarrow\ & p=\pm\sqrt{2}. \end{aligned}$$

In both cases, the value of q is 2.

4664. Let a and b be the first two terms:

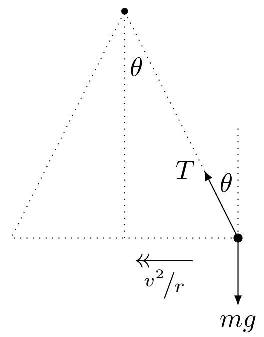
$u_1=a$	$u_4=-a$	$u_7=a$
$u_2=b$	$u_5=-b$	$u_8=b$
$u_3=b-a$	$u_6=a-b$	$u_9=b-a$

The right-hand column is the same as the left-hand column. So, the sequence is certainly periodic, with the greatest possible period being 6. The other possible periods are the factors of 6, which are $1,2,3$:

① for period 1, $a=b=b-a$,
② for period 2, $a=b-a$ and $b=-a$,
③ for period 3, $a=-a$, $b=-b$.

In each case, the only solution to the simultaneous equations is $a=b=0$. So, the sequence is either period 6, or is constant at zero. □

4665. In cross-section, the force diagram is



Resolving horizontally and vertically,

$$\begin{aligned} T\cos\theta &= mg, \\ T\sin\theta &= m\frac{v^2}{r}. \end{aligned}$$

The radius of the circular path is $l\sin\theta$, so the horizontal equation is

$$\begin{aligned} & T\sin\theta = m\frac{v^2}{l\sin\theta} \\ \Longrightarrow\ & T\sin^2\theta = m\frac{v^2}{l} \end{aligned}$$

Dividing the two equations,

$$\begin{aligned} & \frac{\sin^2\theta}{\cos\theta} = \frac{v^2}{gl} \\ \Longrightarrow\ & v^2 = gl\frac{\sin^2\theta}{\cos\theta} \\ \therefore\ & v = \sin\theta\sqrt{\frac{gl}{\cos\theta}}. \end{aligned}$$

The circumference of the circular path is $2\pi l\sin\theta$. So, the time period is

$$\begin{aligned} t &= \frac{2\pi l\sin\theta}{\sin\theta\sqrt{\frac{gl}{\cos\theta}}} \\ &\equiv 2\pi\sqrt{\frac{l\cos\theta}{g}}, \text{ as required.} \end{aligned}$$

4666. Using the factorial definition of ${}^n\mathrm{C}_r$,

$$\begin{aligned} & \mathbb{P}\big(X=3\big) = \mathbb{P}\big(X=4\big) \\ \Longrightarrow\ & \frac{n!}{3!(n-3)!}\tfrac{1}{5}^3\tfrac{4}{5}^{n-3} = \frac{n!}{4!(n-4)!}\tfrac{1}{5}^4\tfrac{4}{5}^{n-4} \\ \Longrightarrow\ & \frac{n(n-1)(n-2)}{6}\cdot 4^{n-3} \\ & \quad = \frac{n(n-1)(n-2)(n-3)}{24}\cdot 4^{n-4}. \end{aligned}$$

The probabilities are not defined for $n=0,1,2$, so we can divide by the relevant factors. This gives

$$\begin{aligned} & \tfrac{1}{6}\cdot 4^{n-3} = \tfrac{n-3}{24}\cdot 4^{n-4} \\ \Longrightarrow\ & 16 = n-3 \\ \Longrightarrow\ & n = 19. \end{aligned}$$

4667. (a) With $\mathrm{f}(x) = x^2$ and $\mathrm{g}(x) = 2x + 1$,

$$\begin{aligned} \mathrm{f} \odot \mathrm{g}(x) &= \mathrm{fg}(x) - \mathrm{gf}(x) \\ &= (2x+1)^2 - (2x^2+1) \\ &\equiv 2x^2 + 4x. \end{aligned}$$

(b) The equation $\mathrm{f} \odot \mathrm{g}(x) = \frac{3\sqrt{3}}{2}$ is

$$\sin 2x - 2\sin x - \tfrac{3\sqrt{3}}{2} = 0.$$

We can't solve directly. But we know that the equation has exactly one root $x \in (-\pi, \pi)$. So, the root must be at a stationary point. Setting the derivative to zero and considering roots in $(-\pi, \pi)$,

$$\begin{aligned} & 2\cos 2x - 2\cos x = 0 \\ \Longrightarrow\ & 2\cos^2 x - \cos x - 1 = 0 \\ \Longrightarrow\ & (2\cos x + 1)(\cos x - 1) = 0 \\ \Longrightarrow\ & x = \pm\tfrac{2\pi}{3}, 0. \end{aligned}$$

Testing these three in the original equation, the required root is $x = -\frac{2\pi}{3}$.

(c) If the function $\mathrm{f} \odot \mathrm{g}$ is identically zero, then the functions fg and gf are identically equal. The order of composition doesn't matter for f and g. In technical language, f and g commute.

— Nota Bene —

The function $\mathrm{f} \odot \mathrm{g}(x) = \mathrm{fg}(x) - \mathrm{gf}(x)$, which is non-zero where f and g fail to commute, is known as the *commutator* of f and g. Such functions play a crucial role in quantum mechanics, where the wavefunctions associated with *fermions* and *bosons* (very broadly, matter and radiation) have different commutation properties.

4668. Assume that $\sqrt{p} + \sqrt{q}$ is rational, and is therefore equal to a/b, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} & \sqrt{p} + \sqrt{q} = \frac{a}{b} \\ \Longrightarrow\ & p + q + 2\sqrt{pq} = \frac{a^2}{b^2} \\ \Longrightarrow\ & \sqrt{pq} = \frac{a^2}{2b^2} - \frac{p+q}{2}. \end{aligned}$$

Since a, b, p, q are integers, the RHS is rational. So, the LHS is rational. Therefore, pq must be a square number. Since p and q are prime, this requires $p = q$. But then

$$\sqrt{p} + \sqrt{q} = 2\sqrt{p}.$$

Since $\sqrt{p}$ is irrational, $2\sqrt{p}$ is also irrational. This is a contradiction. Hence, $\sqrt{p} + \sqrt{q}$ is irrational for primes p and q. □

4669. The coordinates of the relevant point are $\left(\frac{\pi}{4}, 1\right)$. With the line $x = \frac{\pi}{4}$, this forms a rectangle of area $\frac{\pi}{4}$. The area underneath the curve is

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} \tan x \, dx \\ &= \Big[\ln|\sec x| \Big]_0^{\frac{\pi}{4}} \\ &= \ln\left|\sec \tfrac{\pi}{4}\right| - \ln|\sec 0| \\ &= \ln\sqrt{2} - \ln 1 \\ &= \tfrac{1}{4}\ln 4. \end{aligned}$$

So, the area shaded is $\frac{1}{4}\left(\pi - \ln 4\right)$, as required.

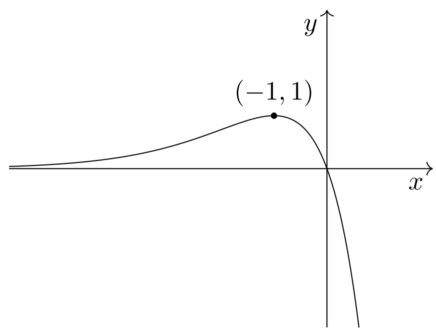
4670. To sketch $y = -xe^{x+1}$, we consider the following:

① Setting $y = 0$, the only axis intercept is O.
② As $x \to \infty$, $y \to -\infty$.
③ As $x \to -\infty$, $y \to 0^+$.
④ Setting the derivative to zero for SPs,

$$\begin{aligned} & -e^{x+1} - xe^{x+1} = 0 \\ \Longrightarrow\ & e^{x+1}(-1 - x) = 0 \\ \Longrightarrow\ & x = -1. \end{aligned}$$

So, there is a stationary point at $(-1, 1)$. Given the behaviour as $x \to -\infty$ and the lack of x intercepts other than at the origin, this must be a maximum.

Putting the above together, the curve is



4671. We assume that "consecutive" is not cyclic, i.e. that A does not follow Z.

There are ${}^{26}\mathrm{C}_3$ ways of selecting three letters from the alphabet. And there are 24 sets of consecutive letters. So, the probability is

$$p = \frac{24}{{}^{26}\mathrm{C}_3} = \tfrac{3}{325}.$$

4672. The terms of the sequence are

1	x_1
2	$ax_1 + b$
3	$a^2x_1 + ab + b$
4	$a^3x_1 + ab^2 + ab + b$
...	...
n	$a^{n-1}x_1 + \underbrace{ab^{n-2} + ab^{n-3} + ... + ab + b}_{*}$.

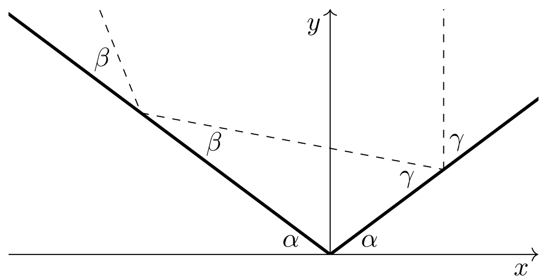
Consider the nth term. Setting aside $a^{n-1}x_1$ at the beginning, the terms notated $*$ form a geometric series. Starting from the end, this has first term b, common ratio a and $n-1$ terms. So,

$$S_{n-1} = \frac{a^{n-1}-1}{a-1}b.$$

Adding the first term back in,

$$x_n = a^{n-1}x_1 + \frac{a^{n-1}-1}{a-1}b, \text{ as required.}$$

4673. The angle α between $y = k|x|$ and the horizontal (x) axis is $\arctan k$. The angle between the ray of incoming light and the horizontal is $\frac{\pi}{2} - \theta$. So, the angle between the incoming light and $y = -kx$ is $\beta = \frac{\pi}{2} - \theta - \arctan k$. This angle is then repeated in the outgoing ray, as shown below.



Angle γ is therefore

$$\begin{aligned}\gamma &= \pi - \beta - (\pi - 2\alpha)\\ &\equiv 2\alpha - \beta\\ &= 2\arctan k - \left(\tfrac{\pi}{2} - \theta - \arctan k\right)\\ &\equiv 3\arctan k - \tfrac{\pi}{2} + \theta.\end{aligned}$$

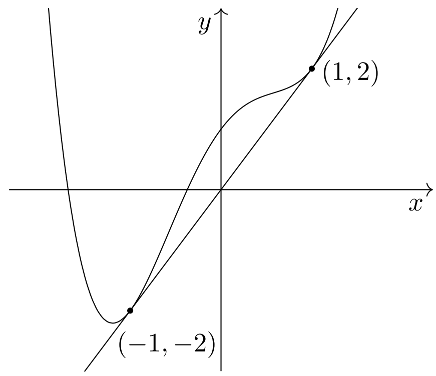
For the outgoing ray to be vertical (parallel to y), we require that $\alpha + \gamma = \frac{\pi}{2}$. This is

$$\begin{aligned}&\arctan k + 3\arctan k - \tfrac{\pi}{2} + \theta = \tfrac{\pi}{2}\\ \Longrightarrow\ &4\arctan k = \pi - \theta\\ \Longrightarrow\ &k = \tan \tfrac{1}{4}(\pi - \theta), \text{ as required.}\end{aligned}$$

4674. Since the line $y = kx$ is tangent to the curve at two distinct points, the equation $x^4 - 2x^2 + 2x + 1 = kx$ must have two double roots:

$$x^4 - 2x^2 + (2-k)x + 1 \equiv (x-a)^2(x-b)^2.$$

The constant term requires $ab = \pm 1$ and the term in x^3 requires $0 = -2a - 2b$. Solving these, $a = 1$ and $b = -1$. So, $k = 2$.



— Alternative Method —

Collecting together the terms of even degree, we can rewrite the equation of the curve as follows:

$$\begin{aligned}y &= x^4 - 2x^2 + 1 + 2x\\ &\equiv (x^2-1)^2 + 2x\\ &\equiv (x+1)^2(x-1)^2 + 2x.\end{aligned}$$

Since $y = (x+1)^2(x-1)^2$ is tangent to the x axis at $x = \pm 1$, the curve in question must be tangent to $y = 2x$ at $x = \pm 1$. So, $k = 2$.

4675. The derivatives ${dy}/{dx}$ and ${dx}/{dy}$ are reciprocals of one another. So, multiplying by ${dy}/{dx}$, this is a quadratic in ${dy}/{dx}$:

$$\begin{aligned}&\left(\frac{dy}{dx}\right)^2 - 3\frac{dy}{dx} - 4 = 0\\ \Longrightarrow\ &\left(\frac{dy}{dx} - 4\right)\left(\frac{dy}{dx} + 1\right) = 0\\ \Longrightarrow\ &\frac{dy}{dx} = 4, -1.\end{aligned}$$

Integrating each alternative individually, either $y = 4x + k_1$ or $y = -x + k_2$, as required.

4676. The gradient is $2a$, so the normal gradient is $-\frac{1}{2a}$. The equation of the normal is therefore

$$y - a^2 = -\tfrac{1}{2a}(x - a).$$

This passes through one of the points $\left(\pm{5}/{\sqrt{8}}, {25}/{8}\right)$. Since $y = x^2$ is symmetrical, we can choose the positive one. Substituting into the above equation,

$$\tfrac{25}{8} - a^2 = -\tfrac{1}{2a}\left(\tfrac{5}{\sqrt{8}} - a\right).$$

This is a cubic in a. Its solution is

$$a \in \left\{-\sqrt{2}, \tfrac{1}{\sqrt{8}}, \tfrac{5}{\sqrt{8}}, \right\}.$$

The last of these is normal to $y = x^2$ <u>at</u> $y = \frac{25}{8}$, so is not an answer we want. The other two are. By symmetry, there are four possible values of a:

$$a \in \left\{\pm\tfrac{1}{\sqrt{8}}, \pm\sqrt{2}\right\}.$$

4677. The equations are a pair of straight lines in the (x, y) plane. These do not have a unique (x, y) solution point if they are parallel. Equating the gradients,

$$\begin{aligned} -\frac{p-1}{q-1} &= -\frac{p^2}{q^2} \\ \Longrightarrow pq^2 - q^2 &= p^2q - p^2 \\ \Longrightarrow pq^2 - p^2q + p^2 - q^2 &= 0. \end{aligned}$$

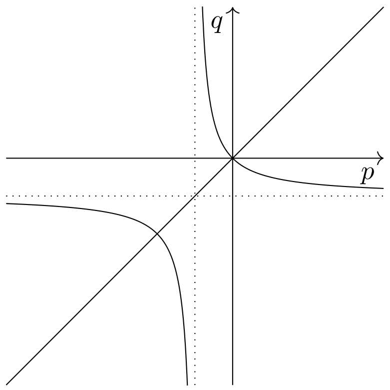
The gradients are clearly equal at $p = q$. So, the above must have a factor of $(p - q)$. Taking it out,

$$\begin{aligned} s(p-q)(pq+p+q) &= 0 \\ \Longrightarrow p = q \quad \text{or} \quad p + pq + q &= 0. \end{aligned}$$

Rearranging the latter,

$$\begin{aligned} q(p+1) &= -p \\ \Longrightarrow q &= \frac{-p}{p+1} = \frac{1}{p+1} - 1. \end{aligned}$$

In (p, q) space, this is a reciprocal graph. It has a vertical asymptote at $p = -1$ and a horizontal asymptote at $q = -1$. Sketching together with $p = q$, the locus of (p, q) values for which the given equations do not have a unique solution is



4678. Converting to a proper fraction, the integrand is

$$1 + \frac{4x+4}{x^2(x+2)}.$$

In partial fractions, the repeated factor x in the denominator necessitates the form

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}.$$

Multiplying up and equating coefficients, we get $A = 1$, $B = 2$ and $C = -1$. We can now integrate:

$$\begin{aligned} &\int_1^2 \frac{x^3 + 2x^2 + 4x + 4}{x^3 + 2x^2}\,dx \\ &= \int_1^2 1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x+2}\,dx \\ &= \Big[x + \ln|x| - 2x^{-1} - \ln|x+2|\Big]_1^2 \\ &= \Big(2 + \ln 2 - 1 - \ln 4\Big) - \Big(1 + \ln 1 - 2 - \ln 3\Big) \\ &= 2 + \ln\tfrac{3}{2}, \text{ as required.} \end{aligned}$$

4679. Solving the boundary equation,

$$\begin{aligned} &\operatorname{cosec}^2 x + \cot x - 1 = 0 \\ \Longrightarrow\ &1 + \cot^2 x + \cot x - 1 = 0 \\ \Longrightarrow\ &\cot x(\cot x + 1) = 0 \\ \Longrightarrow\ &\cot x = 0, -1 \\ \therefore\ \ &x = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{3\pi}{4}, \tfrac{7\pi}{4}. \end{aligned}$$

The vertical asymptotes are at $x = 0, \pi, 2\pi$. These are excluded from the solution set of the inequality. Reading from the graph, the solution set is

$$\Big(0, \tfrac{\pi}{2}\Big] \cup \Big[\tfrac{3\pi}{4}, \pi\Big) \cup \Big(\pi, \tfrac{3\pi}{2}\Big] \cup \Big[\tfrac{7\pi}{4}, 2\pi\Big).$$

4680. The relevant compound-angle formula is

$$\sin(x \pm h) \equiv \sin x \cos h \pm \cos x \sin h.$$

Subtracting the $\pm$ versions of the above, the first terms cancel, giving

$$\sin(x+h) - \sin(x-h) \equiv 2\cos x \sin h.$$

So, the derivative is

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h} \\ &\equiv \lim_{h\to 0} \frac{2\cos x \sin h}{2h} \\ &\equiv \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

It is a standard result that $\displaystyle\lim_{h\to 0} \frac{\sin h}{h} = 1$. So,

$$\frac{dy}{dx} = \cos x, \text{ as required.}$$

— Nota Bene —

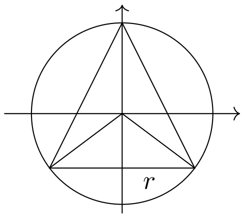
In the usual first-principles differentiation, a chord is drawn between $(x, \sin x)$ and $(x + h, \sin(x + h))$ to the right of it. The latter point is then allowed to tend towards the former.

In this version, we set up $(x - h, \sin(x + h))$ and $(x - h, \sin(x - h))$ to the left and right of $(x, \sin x)$. The chord goes between them, bypassing $(x, \sin x)$. Both points are then allowed to tend towards $(x, \sin x)$. In the case of $y = \sin x$, the algebra is slightly neater this way: the result, of course, is the same!

4681. The derivative is $f'(x) = x^{\frac{1}{2}}$. So, the arc length is

$$\begin{aligned} S &= \int_0^{48} \sqrt{1+x}\,dx \\ &= \Big[\tfrac{2}{3}(1+x)^{\frac{3}{2}}\Big]_0^{48} \\ &= \Big(\tfrac{2}{3} \cdot 49^{\frac{3}{2}}\Big) - \Big(\tfrac{2}{3} \cdot 1^{\frac{3}{2}}\Big) \\ &= 228. \end{aligned}$$

4682. Let the sphere have unit radius. In cross-section, the scenario is



The height of the cone is $1+\sqrt{1-r^2}$. So, the volume of the cone is given by

$$V=\tfrac{1}{3}\pi r^2\left(1+\sqrt{1-r^2}\right).$$

Setting the derivative to zero for optimisation,

$$2r\left(1+\sqrt{1-r^2}\right)+r^2\cdot\tfrac{1}{2}(1-r^2)^{-\frac{1}{2}}\cdot -2r=0.$$

Since $r\neq 0$ (volume minimum),

$$\begin{aligned}&\left(1+\sqrt{1-r^2}\right)-\tfrac{1}{2}r^2(1-r^2)^{-\frac{1}{2}}=0\\ \Longrightarrow\ &\sqrt{1-r^2}+1-r^2-\tfrac{1}{2}r^2=0\\ \Longrightarrow\ &2\sqrt{1-r^2}=3r^2-2\\ \Longrightarrow\ &4-4r^2=9r^4-12r^2+4\\ \Longrightarrow\ &9r^4-8r^2=0\\ \Longrightarrow\ &r=0, r=\sqrt{8/9}.\end{aligned}$$

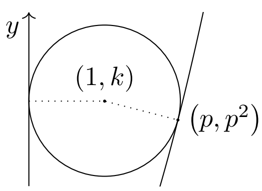
This gives the ratio of volumes as

$$\begin{aligned}\frac{V_{\text{cone}}}{V_{\text{sphere}}}&=\frac{\frac{1}{3}\pi r^2\left(1+\sqrt{1-r^2}\right)}{\frac{4}{3}\pi}\\ &=\frac{\frac{8}{9}(1+\frac{1}{3})}{4}\\ &=\tfrac{8}{27},\text{ as required.}\end{aligned}$$

4683. We integrate by inspection, noting that $\frac{1}{x+1}$ is the derivative of $\ln(x+1)$.

$$\begin{aligned}\mathrm{F}(x)&=\int\frac{1}{(x+1)\ln(x+1)}\,dx\\ &=\ln\big|\ln(x+1)\big|+c.\end{aligned}$$

4684. Since the circle is tangent to the curve, its radius is normal to the curve, with gradient $-\frac{1}{2p}$.



The equation of the radius is

$$y-p^2=-\tfrac{1}{2p}(x-p).$$

Substituting $(1,k)$,

$$\begin{aligned}&k-p^2=-\tfrac{1}{2p}(1-p)\\ \Longrightarrow\ &k=p^2-\tfrac{1}{2p}+\tfrac{1}{2}.\end{aligned}$$

The squared distance from $(1,k)$ to (p,p^2) must be 1. This gives

$$\begin{aligned}&(p-1)^2+\left(-\tfrac{1}{2}+\tfrac{1}{2p}\right)^2=1\\ \Longrightarrow\ &4p^2(p-1)^2+(-p+1)^2-4p^2=0\\ \Longrightarrow\ &4p^4-8p^3+p^2-2p+1=0.\end{aligned}$$

Using a polynomial solver, $p=0.38746$ or 1.9692. The former gives a circle below $y=x^2$, which is tangent to the negative x axis. We reject this root. The latter is the solution: $p=1.969$ (4sf).

4685. Taking out a factor of $(x-y)$, the first equation is

$$(x-y)(x^2+xy+y^2)=\tfrac{7}{16}.$$

Substituting $x-y=1$ and $y=x-1$,

$$\begin{aligned}&x^2+x(x-1)+(x-1)^2=\tfrac{7}{16}\\ \Longrightarrow\ &x=\tfrac{1}{4},\tfrac{3}{4}.\end{aligned}$$

So, the (x,y) solutions are $\left(\frac{1}{4},-\frac{3}{4}\right)$ and $\left(\frac{3}{4},-\frac{1}{4}\right)$.

4686. The integrand is

$$\begin{aligned}\sin^4 x&\equiv\sin^2 x(1-\cos^2 x)\\ &\equiv\sin^2 x-\sin^2 x\cos^2 x.\end{aligned}$$

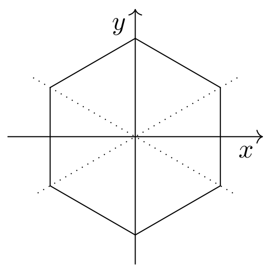
Using double-angle formulae, we write the first term as $\frac{1}{2}(1-\cos 2x)$ and the second term as

$$\tfrac{1}{4}\sin^2 2x\equiv\tfrac{1}{8}(1-\cos 4x).$$

So, the integral is

$$\begin{aligned}&\int_0^\pi \sin^4 x\,dx\\ =\ &\int_0^\pi \tfrac{1}{2}(1-\cos 2x)+\tfrac{1}{8}(1-\cos 4x)\,dx\\ =\ &\int_0^\pi \tfrac{3}{8}-\tfrac{1}{2}\cos 2x-\tfrac{1}{8}\cos 4x\,dx\\ =\ &\left[\tfrac{3}{8}x-\tfrac{1}{4}\sin 2x-\tfrac{1}{32}\sin 4x\right]_0^\pi\\ =\ &\tfrac{3}{8},\text{ as required.}\end{aligned}$$

4687. Other than the points at which the mod functions toggle off/on, the equation generates straight lines. So, the points at which the mod functions toggle are the vertices. These are on $\sqrt{3}x\pm y=0$ and $y=0$, which have rotational symmetry. Each line produces two vertices (one each side of the origin), which generates a regular hexagon.



4688. (a) The area of the shaded region is the sum of the areas of three rectangles:

$$\delta A = h\delta w + w\delta h + \delta w \delta h.$$

Dividing this equation by δt gives

$$\frac{\delta A}{\delta t} = h\frac{\delta w}{\delta t} + w\frac{\delta h}{\delta t} + \frac{\delta w \delta h}{\delta t}.$$

(b) As $\delta t \to 0$, the numerator of the last term tends to zero quadratically (each of $\delta w, \delta h \to 0$ linearly), while the denominator tends to zero linearly. Hence, in the limit as $\delta t \to 0$, the last term becomes 0.

(c) Taking the limit $\delta t \to 0$, the average δ rates of change in the equation in part (a) become instantaneous d rates of change. The last term goes, and we are left with

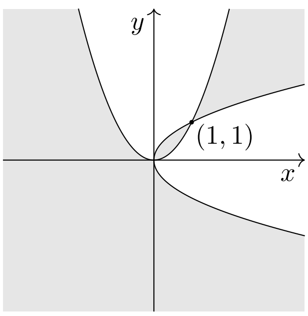
$$\frac{dA}{dt} = h\frac{dw}{dt} + w\frac{dh}{dt}.$$

Since $A = wh$, this is the product rule.

4689. Let $x = a\sin\theta$. This gives $dx = a\cos\theta\, d\theta$. The new limits are $\theta = -\pi/2$ to $\pi/2$.

$$\begin{aligned}
&\int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a\cos\theta\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta \\
&= \Big[\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \pi, \text{ as required.}
\end{aligned}$$

4690. The LHS is a product. It is non-negative iff both of its factors are non-negative or both are negative. The boundary equations are $y = x^2$ and $x = y^2$. The required region is



4691. The general equation of the trajectory of a particle launched from the origin is

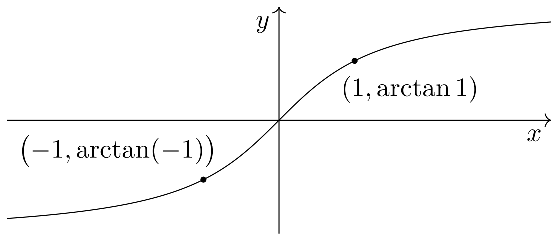
$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}.$$

Equating coefficients, $\tan\theta = 1$, so $\theta = 45°$. This gives $u = \sqrt{32}$. So, horizontal speed is $\sqrt{32}\cos 45°$, which is 4 ms^{-1}. At its highest point, the vertical speed is zero, so the required speed is 4 ms^{-1}.

— Nota Bene —

The equation of the trajectory can be derived by eliminating t from horizontal/vertical *suvat*s.

4692. The range of $\cos x$, for $x \in \mathbb{R}$, is $[-1, 1]$. The graph of $y = \arctan x$, with the endpoints of this domain shown, is



Since $\arctan 1 = \pi/4$, the range of $\arctan(\cos x)$, for $x \in \mathbb{R}$, is $[-\pi/4, \pi/4]$.

4693. The student's logic only implies, in fact, that the oblique asymptote is $y = x + c$. Because there is a $+1$ in both the numerator and denominator, more analysis is required to determine the position of the asymptote. We rewrite as

$$\frac{x^2+1}{x+1} \equiv \frac{x(x+1) - x}{x+1} \equiv x - \frac{x}{x+1}.$$

As $x \to \infty$, the $+1$ in the denominator does now become negligible, and the fraction tends to 1. So, the equation of the asymptote is $y = x - 1$.

4694. Both numerator and denominator have a root at $x = a$, so we cannot yet take the limit safely. As polynomials, they must both have factors of $(x - a)$. Taking this out,

$$\begin{aligned}
&\lim_{x\to a} \frac{x^2 - 3a + 3x - ax}{x^2 - a + x - ax} \\
&\equiv \lim_{x\to a} \frac{(x-a)(x+3)}{(x-a)(x+1)} \\
&\equiv \lim_{x\to a} \frac{x+3}{x+1}.
\end{aligned}$$

We can now take this limit. This gives

$$\begin{aligned}
&\frac{a+3}{a+1} \\
&\equiv \frac{a+1+2}{a+1} \\
&\equiv 1 + \frac{2}{a+1}, \text{ as required.}
\end{aligned}$$

4695. Simplifying,

$$\cos 2x \tan 2x \equiv \cos 2x \frac{\sin 2x}{\cos 2x} \equiv \sin 2x.$$

Differentiating by the chain rule gives $2\cos 2x$.

4696. The derivatives with respect to t are

$$\frac{dx}{dt} = -6\sin 2t,$$
$$\frac{dy}{dt} = 12\cos 2t.$$

Using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12\cos 2t}{-6\sin 2t}\\ &\equiv -2\cot 2t.\end{aligned}$$

Again by the chain rule, the second derivative is

$$\begin{aligned}\frac{d^2y}{dx^2} &\equiv \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &\equiv \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}\\ &= \frac{4\operatorname{cosec}^2 2t}{-6\sin 2t}\\ &\equiv -\frac{2}{3\sin^3 2t}.\end{aligned}$$

Writing this in terms of y,

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{144}{216\sin^3 2t}\\ &= -\frac{144}{y^3}, \text{ as required.}\end{aligned}$$

4697. The middle inequality may be rewritten as

$$\mathbb{P}(X' \cup Y) > \tfrac{3}{4} \iff \mathbb{P}(X \cap Y') < \tfrac{1}{4}.$$

Consider the conditional probability formula

$$\mathbb{P}(X \cap Y') = \mathbb{P}(X) \times \mathbb{P}(Y' \mid X).$$

On the RHS, if $\mathbb{P}(X) > 1/2$ and $\mathbb{P}(Y' \mid X) > 1/2$, then their product is greater than $1/4$. So, on the LHS, $\mathbb{P}(X \cap Y') > 1/4$, which contradicts the earlier result $\mathbb{P}(X \cap Y') < 1/4$. So, the three inequalities cannot be satisfied simultaneously.

4698. This is true. Setting $p = x + y$ and $q = x - y$ gives $p^2 + q^2 = 2$. This is a circle in (p, q) space, whose axes are $y = \pm x$.

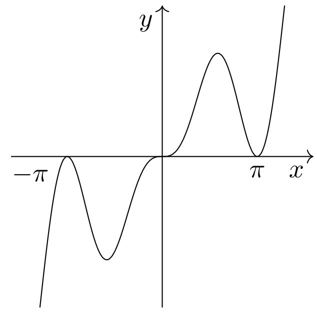
— Nota Bene —

Despite the 2 on the RHS, the equation defines the *unit* circle. That's because the new variables p and q are scaled differently to x and y: the (x, y) point $(1, 1)$ is only $\sqrt{2}$ from the origin, but has $p = 2$.

4699. Using a double-angle identity,

$$\begin{aligned}y &= x(1 - \cos 2x)\\ &\equiv 2x\sin^2 x.\end{aligned}$$

This has the correct behaviour around the origin. However, at the other x intercepts, it is incorrect. The sine function has roots at π and $-\pi$, as shown. But, since the proposed graph contains a *squared* factor $\sin^2 x$, there should be no sign change at these roots. They should be points of tangency with the x axis. The correct graph is



4700. (a) Using the x coordinate,

$$\begin{aligned}\frac{2t}{1+t^2} &= \frac{\sqrt{2}}{2}\\ \implies t &= \sqrt{2} \pm 1.\end{aligned}$$

Substituting in, the root $t = \sqrt{2} - 1$ generates the correct y coordinate. So, P is on the curve.

(b) Using the parametric differentiation formula,

$$\begin{aligned}\frac{dy}{dx} &\equiv \frac{dy/dt}{dx/dt}\\ &= \frac{\frac{2-2t^2}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}}\\ &\equiv \frac{1-t^2}{-2t}.\end{aligned}$$

Evaluating at $t = \sqrt{2} - 1$, the gradient is

$$m = \frac{1 - \left(\sqrt{2}-1\right)^2}{-2\left(\sqrt{2}-1\right)} = -1.$$

So, the tangent at P is $x + y = \sqrt{2}$.

— Alternative Method —

Squaring the equations and adding them,

$$\begin{aligned}x^2 + y^2 &= \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2}\\ &\equiv \frac{1 + 2t^2 + t^4}{(1+t^2)^2}\\ &\equiv \frac{(1+t^2)^2}{(1+t^2)^2}\\ &\equiv 1.\end{aligned}$$

So, the parametric equations define a unit circle centred on the origin.

(a) This follows by Pythagoras.

(b) The tangent is $x + y = \sqrt{2}$.

— End of 47th Hundred —